High-order ADI orthogonal spline collocation method for a new 2D fractional integro-differential problem

Leijie Qiao[a,*], Da Xu[b], Yubin Yan[c]

[a]School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510006, China
[b]Department of Mathematics, Hunan Normal University, Changsha 410081, Hunan, P.R. China
[c]Department of Mathematics, University of Chester, CH1 4BJ, United Kingdom

Abstract

We use the generalized L1 approximation for the Caputo fractional derivative, the second-order fractional quadrature rule approximation for the integral term, and a classical Crank-Nicolson alternating direction implicit (ADI) scheme for the time discretization of a new two-dimensional (2D) fractional integro-differential equation, in combination with a space discretization by an arbitrary-order orthogonal spline collocation (OSC) method. The stability of a Crank-Nicolson ADI OSC scheme is rigourously established, and error estimate is also derived. Finally, some numerical tests are given.

Keywords: two-dimensional fractional integro-differential equation, Crank-Nicolson alternating direction implicit scheme, orthogonal spline collocation method, stability, convergence

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1. Introduction

We consider a high-order Crank-Nicolson ADI OSC for the following 2D initial-boundary value fractional integro-differential problem

$$D_t^{\alpha}u(x,y,t)-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\Delta u(x,y,s)ds=\lambda\Delta u(x,y,t)+f(x,y,t),\quad (x,y)\in\Omega,\quad t\in(0,T],\tag{1.1}$$

$$u(x,y,0)=\phi(x,y),\quad u_t(x,y,0)=\psi(x,y),\quad (x,y)\in\overline{\Omega}=\Omega\cup\partial\Omega,\tag{1.2}$$

$$u(x,y,t)=0,\quad (x,y)\in\partial\Omega,\quad t\in(0,T],\tag{1.3}$$

where $\Omega=(0,\ L^x)\times(0,\ L^y)(L^x,L^y>0)$ with a smooth boundary $\partial\Omega$, Δ is the Laplace operator, $\phi(x,y)$, $\psi(x,y)$ and $f(x,y,t)$ are given smooth functions, $\lambda>0$. The integral term is the Riemann-Liouville fractional integration operator of order $\beta(0<\beta<1)$. The symbol D_t^{α} in (1.1) denotes the Caputo fractional derivative

$$D_t^{\alpha}u(x,y,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{\partial^2u(x,y,s)}{\partial s^2}\frac{ds}{(t-s)^{\alpha-1}},\ (x,y)\in\Omega,\ t\in(0,T],\ 1<\alpha<2.$$

The modeling of real-world phenomena is becoming a popular topic in the past few decades. Mathematical models of real-world phenomena are often associated with fractional calculus. In particular,

*Corresponding author

Email addresses: qiaoleijie@yeah.net (Leijie Qiao), daxu@hunnu.edu.cn (Da Xu), y.yan@chester.ac.uk (Yubin Yan)

physical scientists and mathematicians, since it can faithfully capture the dynamics of physical process in many applied sciences including ecology, biology, control system.

We can see that the problem (1.1) is very interesting and easily obtain the following 2D fractional evolution equation if $\alpha = 1$ in (1.1)

$$u_t(x,y,t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Delta u(x,y,s)ds = \lambda\Delta u(x,y,t) + f(x,y,t),\ (x,y)\in\Omega,\ t\in(0,T]. \quad (1.4)$$

The fractional evolution equations of (1.4) with different boundary and initial conditions often occur in many applications such as heat conduction in material with memory, compression of viscoelastic media, population dynamics, nuclear reactor dynamics [1], etc. The excellent literatures on the numerical solutions for (1.4) is very rich. We just mention a few part among them. Yang. et al [9] constructed a scheme by using the OSC method in space and the finite difference scheme in time for (1.4) with $\lambda = 0$. K. Mustapha and H. Mustapha [3] used the linear finite element method for semilinear problem of (1.4). Xu [7, 8] applied the finite element method to (1.4) with $\lambda > 0$. Fairweather et al. [11][12] used the ADI OSC method for the problem similar to (1.4) with smooth index kernel. Qiao and Xu [2] proposed an ADI compact finite difference numerical method for (1.4) with $\lambda > 0$. Yi and Guo [31] used Petrov-Galerkin finite element method for Volterra integro-differential equations with smooth and non-smooth kernels similar to (1.4). Wang et al. [32] considered the Legendre-Jacobi spectral collocation method for the Volterra integro-differential equations with smooth and weakly singular kernels.

If there is no integral term in (1.1), we get the following familiar 2D diffusion-wave equation

$$D_t^\alpha u(x,y,t) = \lambda\Delta u(x,y,t) + f(x,y,t),\ \ (x,y)\in\Omega,\ \ t\in(0,T]. \quad (1.5)$$

The problems of this type with different initial-boundary values applied in the modeling of anomalous diffusive and sub-diffusive systems, the description of fractional random walk, the unification of diffusion and wave propagation phenomena [20, 21], etc. Over the past several years, there are massive literatures using different numerical methods and techniques for the analytical solutions and other properties to (1.5). For example, Zhang and Ren et al. [22] studied compact finite difference method for (1.5). Fairweather et al. [24] proposed an ADI OSC numerical method for (1.5).

If $\alpha = 2$ in (1.1), another important problem related to (1.1) is the following hyperbolic integro-differential problem

$$u_{tt}(x,t) + \int_0^t e^{-\eta(t-s)}u_{xx}(x,s)ds = \lambda u_{xx}(x,t) + f(x,t),\ x\in\Omega,\ t\in(0,T], \quad (1.6)$$

Xu [26] studied the problem of boundary observability with OSC method for (1.6). P. Loreti and D. Sforza [28, 29] used harmonic approach analyzed (1.6).

We have proposed an ADI finite difference scheme similar to problem (1.1) with $0 < \alpha < 1$, $0 < \beta < 1$ in [30], but limited to the low-order approximations in space direction. In this paper, we give a high-order Crank-Nicolson ADI OSC numerical scheme for (1.1). The innovations of this paper are mainly in three aspects: (1) To the best of author's knowledge, we have not, as yet, found any reports on the numerical methods for solving (1.1). (2) We proposed a Crank-Nicolson ADI OSC scheme for the numerical solution of (1.1). Meanwhile we gave numerical examples to verify the effectiveness of the proposed numerical scheme. It is the first time that such a class of problems have been tackled with a Crank-Nicolson ADI OSC scheme. (3) We carried out a rigorously analysis of stability and convergence of the proposed Crank-Nicolson ADI OSC method. This task is not trivial since the Caputo derivative and integral term occur simultaneously in (1.1), as we will see later, that the theoretical analysis for $1 < \alpha < 2$ are complicated and different from that of $0 < \alpha < 1$ in [30].

We are interested in an ADI scheme since it is a highly efficient procedures for solving parabolic and hyperbolic initial-boundary value problems [14, 25, 18, 10], it reduces the solution of the multi-dimensional problem to solve a series of similar, independent one-dimensional problems, thus reducing the computational cost. Furthermore, since the differential operator in (1.1) has constant coefficients, the coefficient matrices in each coordinate direction are independent of time and can be decomposed only once for the entire problem. This effective technique in conjunction with various types of problems has been studied extensively in the literatures. See, for instance, Gao and Sun [13] considered an ADI finite difference scheme for 2D distributed-order fractional diffusion equations. Yu et al. [14] constructed an ADI finite difference method for the fractional Bloch-Torrey equations to studied anomalous diffusion in the human brain. As point out in [25, 19], the well-known advantage of OSC procedures over Galerkin finite element procedures is that the formation of the coefficients in the equations which determined the numerical solution is very fast since no integrals need to be evaluated. Also, unlike finite difference methods, it yields continuous approximations to the solution and its first derivatives of high-order accuracy throughout. Based on the excellent features of an ADI OSC method described above, we propose a Crank-Nicolson ADI OSC numerical scheme for (1.1).

An outline of the paper is as follows. In Section 2, we introduce some notations and auxiliary lemmas. Meanwhile, a Crank-Nicolson ADI OSC method is formulated. In Section 3, the convergence result is presented. Finally, in Section 4, we present several numerical experiments which confirms the analytical rates of convergence.

2. The proposed Crank-Nicolson ADI OSC

2.1. Preliminaries

In this section, we introduce some useful notations. For a bounded domain $\Omega \subset \mathbb{R}^2$, let L^m, $1 \leq m \leq \infty$, be the Banach space with norm $\|\cdot\|_{L^m}$, where

$$\|u\|_{L^m} = \left(\int_\Omega |u|^m dxdy\right)^{1/m}, \quad \|u\|_{L^\infty} = \sup_\Omega |u|.$$

For $m > 0$, the $H^m(\Omega)$ norm on Sobolev space defined by

$$\|u\|_{H^m} = \left(\sum_{0\leq\alpha_1+\alpha_2\leq m} \left\|\frac{\partial^{\alpha_1+\alpha_2}u}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\right\|^2\right)^{\frac{1}{2}}.$$

Let $\delta_x = \{x_i\}_{i=0}^{N_x}$ and $\delta_y = \{y_j\}_{j=0}^{N_y}$ be two partitions of $\overline{I^x} = [0, L^x]$ and $\overline{I^y} = [0, L^y]$, respectively, such that

$$0 = x_0 < x_1 < \cdots < x_{N_x} = L^x, \quad 0 = y_0 < y_1 < \cdots < y_{N_y} = L^y.$$

$\delta = \delta_x \times \delta_y$ denotes the partition of Ω is quasi-uniform. Set

$$h(\delta_x) = \max_{1\leq i\leq N_x} \{x_i - x_{i-1}\}, \quad h(\delta_y) = \max_{1\leq j\leq N_y} \{y_j - y_{j-1}\}, \quad h = \max\{h(\delta_x), h(\delta_y)\}.$$

Make $P(r, \delta_x)$ be the spaces of piecewise polynomials of degree $\leq r (r \geq 3)$, defined by

$$P(r, \delta_x) = \{v | v \in C^1((0, L^x]),\ v \in P_r([x_{i-1}, x_i]),\ i = 1, 2, \ldots, N_x,\ v(0) = v(L^x) = 0\},$$

where P_r denotes the set of polynomials of degree at most r. With $P(r, \delta_y)$ and $h(\delta_y)$ defined similarly.

Then let

$$P_r(\delta) = P(r,\delta_x) \otimes P(r,\delta_y),$$

be the set of all functions that are linear combinations of products of the form $\upsilon_1\upsilon_2$, where $\upsilon_1 \in P(r,\delta_x)$ and $\upsilon_2 \in P(r,\delta_y)$.

Suppose $\Gamma = \{(\xi_{l,i}(\delta_x), \xi_{k,j}(\delta_y)) | 1 \le \{l,k\} \le r-1,\ 1 \le i \le N_x,\ 1 \le j \le N_y\}$ be the set of Gauss points in Ω. $\{\gamma_k\}_{k=1}^{r-1}$ and $\{\gamma_l\}_{l=1}^{r-1}$ be the weights of the Gaussian quadrature rule. For u and v defined on Γ, the discrete inner product and the norm are represented by

$$\langle u, \upsilon \rangle = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(x_i - x_{i-1})(y_j - y_{j-1})\sum_{l=1}^{r-1}\sum_{k=1}^{r-1}\gamma_l\gamma_k(u\upsilon)(\xi_{l,i}(\delta_x), \xi_{k,j}(\delta_y)),$$

and

$$|\upsilon|_D^2 = \langle \upsilon, \upsilon \rangle.$$

2.2. Derivation of a Crank-Nicolson ADI OSC scheme

To describe our fully discrete Crank-Nicolson ADI OSC method, we set $\{t_n\}_{n=0}^N$ is a uniform partition of $[0,T]$ such that $t_n = n\tau,\ t_{n-\frac{1}{2}} = (n-\frac{1}{2})\tau,\ \tau = \frac{T}{N}$. We shall often also use the analogous notations

$$u^n = u(\cdot, t_n),\quad \delta_t u^n = \frac{1}{\tau}(u^n - u^{n-1}),\quad u^{n-\frac{1}{2}} = \frac{1}{2}(u^n + u^{n-1}),\quad f^n = f(\cdot, t_n).$$

We now approximate Caputo derivative in (1.1) at $t_{n-\frac{1}{2}} (1 \le n \le N)$ using the result in [27, 16] such that

$$\begin{aligned} D_t^\alpha u(x,y,t_{n-\frac{1}{2}}) =& \frac{1}{\Gamma(2-\alpha)}\int_0^{t_{n-\frac{1}{2}}} \frac{\partial^2 u(x,y,s)}{\partial s^2}\frac{ds}{(t_{n-\frac{1}{2}} - s)^{\alpha-1}} \\ =& \frac{1}{\Gamma(3-\alpha)}\frac{1}{\tau^{\alpha-1}}\left[b_0\delta_t u(x,y,t_n) - \sum_{j=1}^{n-1}(b_{n-j-1} - b_{n-j})\delta_t u(x,y,t_j) - b_{n-1}\psi(x,y)\right] \\ &+ (R_t^\alpha)^{n-\frac{1}{2}}, \end{aligned} \tag{2.1}$$

with $b_j = (j+1)^{2-\alpha} - j^{2-\alpha},\ j \ge 0$, and the truncation error $(R_t^\alpha)^{n-\frac{1}{2}}$ in (2.1) is

$$|(R_t^\alpha)^{n-\frac{1}{2}}| \le C \max_{0 \le t \le T} |u_{tt}(x,y,t)|\tau^{3-\alpha},\quad 1 \le\ n \le N. \tag{2.2}$$

Then, we give the main result from Lubich [4, 5, 6] to approximate the convolution integral term in (1.1)

$$q_n(\varphi) = \tau^\beta \sum_{p=0}^{n}\gamma_{n-p}\varphi^p + \rho_n\varphi^0, \tag{2.3}$$

in which the quadrature weights γ_p are determined by the generating power series

$$\left[\frac{3}{2} - 2z + \frac{1}{2}z^2\right]^{-\beta} = \sum_{p=0}^{\infty}\gamma_p z^p.$$

We use the correction quadrature weights ρ_n thus the quadrature formula becomes exact for constant $\varphi = 1$

$$\tau^\beta \sum_{p=0}^{n}\gamma_p + \rho_n = \frac{1}{\Gamma(\beta)}\int_0^{t_n}(t_n - s)^{\beta-1}ds = \frac{1}{\Gamma(\beta+1)}t_n^\beta \le C(T). \tag{2.4}$$

From (2.3) and (2.4), we obtain the second-order convolution quadrature scheme at $t_{n-\frac{1}{2}}$

$$q_{n-\frac{1}{2}}(\varphi)=\tau^{\beta}\sum_{p=0}^{n}\gamma_{n-p}\varphi^{p-\frac{1}{2}}+\rho_{n-\frac{1}{2}}\varphi^{0}, \tag{2.5}$$

where

$$\varphi^{-\frac{1}{2}}=\frac{\varphi^{0}}{2},\ \ \varphi^{-1}=0.$$

The bound of $\varepsilon(\varphi)(t_n)=\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds-q_n(\varphi)$ follows from the next lemma.

Lemma 2.1. [8] *Let φ be a real, continuously differentiable function on $(0,T]$. Then, the error of fractional quadrature rule is proposed by*

$$|\varepsilon(\varphi)(t_n)|\leq C\tau^2t_n^{\beta-1}|\varphi_t(0)|+C\tau^{\beta+1}\int_{t_{n-1}}^{t_n}|\varphi_{tt}(s)|ds+C\tau^2\int_0^{t_{n-1}}\frac{(t_n-s)^{\beta-1}}{\Gamma(\beta)}|\varphi_{tt}(s)|ds,\ \ n\geq 1.$$

Lemma 2.2. [17] *It is important to note that $|\cdot|_D$ and $\|\cdot\|$ are equivalent on $P_r(\delta)$.*

According to the approximation of Caputo derivative given by (2.1) and the convolution quadrature scheme proposed by (2.5), with the notations $U^{n-\frac{1}{2}}=\frac{1}{2}(U^n+U^{n-1}),\ f^{n-\frac{1}{2}}=\frac{1}{2}(f^n+f^{n-1})$, thus the Crank-Nicolson OSC scheme for the approximation of (1.1) consists in finding $U^n\in P_r(\delta),\ n=1,2,\dots,N$, such that

$$\begin{aligned}&\frac{1}{\Gamma(3-\alpha)}\frac{1}{\tau^{\alpha-1}}\left[b_0\delta_tU^n-\sum_{j=1}^{n-1}(b_{n-j-1}-b_{n-j})\delta_tU^j-b_{n-1}\psi\right]-\mu\Delta U^{n-\frac{1}{2}}\\&=\tau^{\beta}\sum_{p=0}^{n}\gamma_{n-p}\Delta U^{p-\frac{1}{2}}+\rho_{n-\frac{1}{2}}\Delta U^0+f^{n-\frac{1}{2}}\ \text{ on }\ \Gamma.\end{aligned} \tag{2.6}$$

In order to achieve an ADI scheme, we need to add a small term

$$\frac{\lambda_1\tau(\mu+\lambda_2)^2}{4}\frac{\partial^4\delta_tU^n}{\partial x^2\partial y^2},$$

to the left-hand side of (2.6), then we obtain

$$\begin{aligned}&\frac{1}{\Gamma(3-\alpha)}\frac{1}{\tau^{\alpha-1}}\left[b_0\delta_tU^n-\sum_{j=1}^{n-1}(b_{n-j-1}-b_{n-j})\delta_tU^j-b_{n-1}\psi\right]-\mu\Delta U^{n-\frac{1}{2}}+\frac{\lambda_1\tau(\mu+\lambda_2)^2}{4}\frac{\partial^4\delta_tU^n}{\partial x^2\partial y^2}\\&=\tau^{\beta}\sum_{p=0}^{n}\gamma_{n-p}\Delta U^{p-\frac{1}{2}}+\rho_{n-\frac{1}{2}}\Delta U^0+f^{n-\frac{1}{2}}\ \text{ on }\ \Gamma,\end{aligned} \tag{2.7}$$

where $\lambda_1=\tau^{\alpha}\Gamma(3-\alpha),\ \lambda_2=\tau^{\beta}\gamma_0,\ b_0=1$. For $p=0$ in (2.7), we define

$$\Delta U^{-\frac{1}{2}}=\frac{1}{2}\Delta U^0.$$

With $E^n=U^n-U^{n-1}$, multiplying (2.7) by λ_1 and rearranging terms, we rewrite (2.7) in the following form

$$\left[1-\left(\frac{\lambda_1\mu}{2}+\frac{\lambda_1\lambda_2}{2}\right)\Delta+\frac{(\lambda_1\mu+\lambda_1\lambda_2)^2}{4}\frac{\partial^4}{\partial x^2\partial y^2}\right]E^n=F^n\ \text{ on }\ \Gamma,\ \ 1\leq n\leq N, \tag{2.8}$$

where

$$\begin{aligned}F^n =&\lambda_1\tau^\beta\sum_{p=0}^{n-1}\gamma_{n-p}\Delta U^{p-\frac{1}{2}}+\sum_{j=1}^{n-1}(b_{n-j-1}-b_{n-j})E^j-\tau b_{n-1}\psi+\lambda_1\rho_{n-\frac{1}{2}}\Delta U^0\\&+(\mu\lambda_1+\lambda_1\lambda_2)\Delta U^n+\lambda_1 f^{n-\frac{1}{2}}.\end{aligned}$$

We now rewrite (2.8) in a Crank-Nicolson ADI OSC scheme. To this end, let $\{\varphi_i\}_{i=1}^{M_x}$ and $\{\psi_j\}_{j=1}^{M_y}$ be bases for the subspace $P(r,\delta_x)$ and $P(r,\delta_y)$, respectively, where $M_x=(r-1)N_x$ and $M_y=(r-1)N_y$, denote

$$U^n(x,y)=\sum_{i=1}^{M_x}\sum_{j=1}^{M_y}\omega_{ij}^{(n)}\varphi_i(x)\psi_j(y),$$

and let

$$\omega^{(n)}=[\omega_{11}^{(n)},\omega_{12}^{(n)},\ldots,\omega_{1M_y}^{(n)},\omega_{21}^{(n)},\ldots,\omega_{M_xM_y}^{(n)}]^T.$$

Here and below we set the matrices

$$\begin{aligned}A_x&=\{(\omega_{ij}^x)_{i,j=1}^{M_x},\omega_{ij}^x=-\varphi_j^{''}(\xi_i(\delta_x))\},\ \ B_x=\{(b_{ij}^x)_{i,j=1}^{M_x},b_{ij}^x=\varphi_j(\xi_i(\delta_x))\},\\A_y&=\{(\omega_{ij}^y)_{i,j=1}^{M_y},\omega_{ij}^y=-\psi_j^{''}(\xi_i(\delta_y))\},\ \ B_y=\{(b_{ij}^y)_{i,j=1}^{M_y},b_{ij}^y=\varphi_j(\xi_i(\delta_y))\},\end{aligned}$$

and

$$F^{(n)}=[F^{(n)}(\xi_1(\delta_x),\xi_1(\delta_y))\ldots,F^{(n)}(\xi_1(\delta_x),\xi_{M_y}(\delta_y)),F^{(n)}(\xi_2(\delta_x),\xi_{M_y}(\delta_y)),\ldots,F^{(n)}(\xi_{M_x}(\delta_x),\xi_{M_y}(\delta_y))]^T.$$

We rewrite (2.8) as

$$\left[B_x\otimes B_y+\frac{(\lambda_1\mu+\lambda_1\lambda_2)}{2}(A_x\otimes B_y+B_x\otimes A_y)+\frac{(\lambda_1\mu+\lambda_1\lambda_2)^2}{4}A_x\otimes A_y\right]E^{(n)}=F^{(n)},\tag{2.9}$$

where $\otimes$ denotes the matrix tensor product. (2.9) is equivalent to the following Crank-Nicolson ADI OSC scheme

$$\begin{aligned}&\left[\left(B_x+\frac{(\lambda_1\mu+\lambda_1\lambda_2)}{2}A_x\right)\otimes I_{M_y}\right]\widetilde{\upsilon}^{(n)}=F^{(n)},\\&\left[I_{M_x}\otimes\left(B_y+\frac{(\lambda_1\mu+\lambda_1\lambda_2)}{2}A_y\right)\right]\widetilde{\omega}^{(n)}=\widetilde{\upsilon}^{(n)},\ 1\le n\le N,\end{aligned}$$

where

$$\widetilde{\omega}^{(n)}=\omega^{(n)}-\omega^{(n-1)}.$$

Thus we compute $\widetilde{\omega}^{(n)}$ by solving two sets of independent one-dimensional problem, first

$$\left(B_x+\frac{(\lambda_1\mu+\lambda_1\lambda_2)}{2}A_x\right)\widetilde{\upsilon}_j^{(n)}=F_j^{(n)},\ j=1,2,\ldots,M_y,\ 1\le n\le N,$$

in the x-direction, where $\widetilde{\upsilon}_j^{(n)}=[\widetilde{\upsilon}_{1j}^{(n)},\widetilde{\upsilon}_{2j}^{(n)},\ldots,\widetilde{\upsilon}_{M_xj}^{(n)}]^T$, followed by

$$\left(B_y+\frac{(\lambda_1\mu+\lambda_1\lambda_2)}{2}A_y\right)\widetilde{\omega}^{(n)}=\widetilde{\upsilon}_i^{(n)},\ i=1,2,\ldots,M_x,\ 1\le n\le N,$$

in the y-direction, where $\widetilde{\omega}_i^{(n)}=[\widetilde{\omega}_{i1}^{(n)},\widetilde{\omega}_{i2}^{(n)},\ldots,\widetilde{\omega}_{iM_y}^{(n)}]^T$.
Then

$$\omega^{(n)}=\widetilde{\omega}^{(n)}+\omega^{(n-1)}.$$

3. Stability of the Crank-Nicolson ADI OSC scheme

Throughout this paper, the generic constant C independent of essential quantities and not necessarily the same at each occurrence. In the stability analysis, we shall often repeat use the Young's inequality

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \ \ a, b \in \mathbb{R}, \ \ \varepsilon > 0,$$

and Minkowski inequality

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)},$$

where $1 \le p \le \infty, f, g \in L^p$. The proof of the stability analysis relies on the following lemmas.

Lemma 3.1. *Assume $\nu, \upsilon \in P_r(\delta)$, we have*

$$([25], (3.4)); \ \langle -\Delta\nu, \upsilon\rangle = \langle \nu, -\Delta\upsilon\rangle,$$

$$([25], (3.5)); \ \langle -\Delta\nu, \nu\rangle \ge \mathrm{C}\|\nabla\nu\|^2 \ge 0,$$

$$([25], \text{Lemma 3.3}), \ |\langle \Delta\nu, \upsilon\rangle| \le \mathrm{C}\|\nabla\nu\|\|\nabla\upsilon\|, \ \ -\langle \Delta\nu, \nu\rangle \le \mathrm{C}[\|\nabla\nu\|^2 + \|\nabla\upsilon\|^2].$$

Lemma 3.2. *For any $U = \{U^0, U^1, \dots, U^{n-1}, U^n\}$ on Γ, the coefficients c_i have the following property* [27, 16]

$$\sum_{n=1}^{N}\left[c_0|U^n|_D - \sum_{k=1}^{n-1}(c_{n-k-1}-c_{n-k})|U^k|_D - c_{n-1}|U^0|_D\right]|U^n|_D \ge \frac{t_N^{1-\alpha}}{2}\tau\sum_{n=1}^{N}|U^n|_D^2 - \frac{t_N^{2-\alpha}}{2(2-\alpha)}|U^0|_D^2,$$

where $1 < \alpha < 2$, $U^j \in P_r(\delta)$, $c_i = \frac{\tau^{2-\alpha}}{2-\alpha} b_i$, $i, j = 0, 1, \dots, n-1, n$, b_i is defined in (2.1).

Theorem 3.1. *The Crank-Nicolson ADI OSC scheme (2.8) is stable with respect to the H^1 norm for $\tau < \frac{\mu}{4C(T)}$, $\mu > 0$. Specifically, for $U^m \in P_r(\delta), 1 \le m \le N$,*

$$\|\nabla U^m\|^2 \le C\|\nabla U^0\|^2 + C\tau\|\nabla U^0\|^2 + C\tau^2\|\nabla U^0\|^2 + C(T)\tau\sum_{n=1}^{m}|f^{n-\frac{1}{2}}|_D^2 + Ct_N^{2-\alpha}|\psi|_D^2.$$

Proof. From (2.7), for $1 \le n \le N$, we have

$$\begin{aligned}&\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left[c_0\delta_t U^n - \sum_{j=1}^{n-1}(c_{n-j-1}-c_{n-j})\delta_t U^j - c_{n-1}\psi\right] - \mu\Delta U^{n-\frac{1}{2}} + \frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\frac{\partial^4\delta_t U^n}{\partial x^2\partial y^2}\\ =&\tau^\beta\sum_{p=0}^{n}\gamma_{n-p}\Delta U^{p-\frac{1}{2}} + \rho_{n-\frac{1}{2}}\Delta U^0 + f^{n-\frac{1}{2}} \ \text{ on } \ \Gamma.\end{aligned} \tag{3.1}$$

For all $\upsilon \in P_r(\delta)$, $1 \le n \le N$, (3.1) is equivalent to

$$\begin{aligned}&\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left[c_0\langle\delta_t U^n, \upsilon\rangle - \sum_{j=1}^{n-1}(c_{n-j-1}-c_{n-j})\langle\delta_t U^j, \upsilon\rangle - c_{n-1}\langle\psi, \upsilon\rangle\right] - \mu\langle\Delta U^{n-\frac{1}{2}}, \upsilon\rangle\\ &+\frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\left\langle\frac{\partial^4\delta_t U^n}{\partial x^2\partial y^2}, \upsilon\right\rangle = \tau^\beta\sum_{p=0}^{n}\gamma_{n-p}\langle\Delta U^{p-\frac{1}{2}}, \upsilon\rangle + \rho_{n-\frac{1}{2}}\langle\Delta U^0, \upsilon\rangle + \langle f^{n-\frac{1}{2}}, \upsilon\rangle \ \text{ on } \Gamma.\end{aligned} \tag{3.2}$$

Then, selecting $v = \delta_t U^n$ in (3.2), we have

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left[c_0\langle\delta_t U^n,\delta_t U^n\rangle-\sum_{j=1}^{n-1}(c_{n-j-1}-c_{n-j})\langle\delta_t U^j,\delta_t U^n\rangle-c_{n-1}\langle\psi,\delta_t U^n\rangle\right]-\mu\langle\Delta U^{n-\frac{1}{2}},\delta_t U^n\rangle\\ &+\frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\left\langle\frac{\partial^4\delta_t U^n}{\partial x^2\partial y^2},\delta_t U^n\right\rangle \\ =&\tau^\beta\sum_{p=0}^{n}\gamma_{n-p}\langle\Delta U^{p-\frac{1}{2}},\delta_t U^n\rangle+\rho_{n-\frac{1}{2}}\langle\Delta U^0,\delta_t U^n\rangle+\langle f^{n-\frac{1}{2}},\delta_t U^n\rangle \ \text{ on } \Gamma. \end{aligned} \tag{3.3}$$

We bound the third term on the left-hand side of (3.3) using the inequality in ([12], Lemma 3.4) that

$$\frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\left\langle\frac{\partial^4\delta_t U^n}{\partial x^2\partial y^2},\delta_t U^n\right\rangle\geq\frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\left\|\frac{\partial^2\delta_t U^n}{\partial x\partial y}\right\|^2\geq 0,\ \ 1\leq n\leq N. \tag{3.4}$$

Using the results in [12], we obtain

$$-\mu\langle\Delta U^{n-\frac{1}{2}},\delta_t U^n\rangle=\frac{\mu}{2}\delta_t\langle-\Delta U^n,U^n\rangle. \tag{3.5}$$

Now, for $2\leq n\leq N$, using the definition of $q_n(\varphi)$ in (2.3), the first and second terms on the right-hand side of (3.3) can be rewritten as

$$\begin{aligned} \langle q_{n-\frac{1}{2}}(\Delta U),\delta_t U^n\rangle=&\delta_t\langle q_{n-\frac{1}{2}}(\Delta U),U^n\rangle-\langle\delta_t\rho_{n-\frac{1}{2}}\Delta U^0,U^{n-1}\rangle-\tau^\beta\sum_{p=0}^{n-1}\delta_t\gamma_{n-p}\langle\Delta U^{p-\frac{1}{2}},U^{n-1}\rangle\\ &-\gamma_0\tau^{\beta-1}\langle\Delta U^{n-\frac{1}{2}},U^{n-1}\rangle. \end{aligned} \tag{3.6}$$

On substituting (3.4)-(3.6) into (3.3), multiplying the resulting equation by $\tau\Gamma(2-\alpha)$, then summing resulting expression from $n=1$ to $n=m$, $1\leq m\leq N$, we obtain

$$\begin{aligned} &\sum_{n=1}^{m}\left[c_0\langle\delta_t U^n,\delta_t U^n\rangle-\sum_{j=1}^{n-1}(c_{n-j-1}-c_{n-j})\langle\delta_t U^j,\delta_t U^n\rangle-c_{n-1}\langle\psi,\delta_t U^n\rangle\right]\\ &-\frac{\mu\tau\Gamma(2-\alpha)}{2}\sum_{n=1}^{m}\delta_t\langle\Delta U^n,U^n\rangle+\frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\sum_{n=1}^{m}\left\|\frac{\partial^2\delta_t U^n}{\partial x\partial y}\right\|^2\\ \leq&\Gamma(2-\alpha)\left[|\langle q_{m-\frac{1}{2}}(\Delta U),U^m\rangle|+|\langle q_{\frac{1}{2}}(\Delta U),U^0\rangle|\right]\\ &+\tau^{\beta+1}\Gamma(2-\alpha)\sum_{n=1}^{m}\sum_{p=0}^{n-1}|\delta_t\gamma_{n-p-\frac{1}{2}}||\langle\Delta U^p,U^{n-1}\rangle|+\Gamma(2-\alpha)\tau^\beta|\gamma_0|\sum_{n=1}^{m}|\langle\Delta U^{n-\frac{1}{2}},U^{n-1}\rangle|\\ &+\tau\Gamma(2-\alpha)\left|\sum_{n=1}^{m}\delta_t\rho_{n-\frac{1}{2}}\langle\Delta U^0,U^{n-1}\rangle\right|+\tau\Gamma(2-\alpha)\sum_{n=1}^{m}|\langle f^{n-\frac{1}{2}},\delta_t U^n\rangle|. \end{aligned} \tag{3.7}$$

Using Lemma 3.2, the first term on the left-hand side of (3.7) may be bounded by

$$\begin{aligned} &\sum_{n=1}^{m}\left[c_0\langle\delta_t U^n,\delta_t U^n\rangle-\sum_{j=1}^{n-1}(c_{n-j-1}-c_{n-j})\langle\delta_t U^j,\delta_t U^n\rangle-c_{n-1}\langle\psi,\delta_t U^n\rangle\right]\\ \geq&\frac{t_N^{1-\alpha}}{2}\tau\sum_{n=1}^{m}|\delta_t U^n|_D^2-\frac{t_N^{2-\alpha}}{2(2-\alpha)}|\psi|_D^2, \end{aligned} \tag{3.8}$$

For the second term on the left-hand side of (3.7), using Lemma 3.1, we easily have

$$
\begin{aligned}
-\frac{\mu\tau\Gamma(2-\alpha)}{2}\sum_{n=1}^{m}\delta_t\langle\Delta U^n,U^n\rangle &= -\frac{\mu\Gamma(2-\alpha)}{2}[\langle\Delta U^m,U^m\rangle-\langle\Delta U^0,U^0\rangle]\\
&\geq\frac{\mu\Gamma(2-\alpha)}{2}[C\|\nabla U^m\|^2+\langle\Delta U^0,U^0\rangle].
\end{aligned}\tag{3.9}
$$

Applying the Lemma 3.1 and Young's inequality to the last term on the right-hand side of (3.7), we have

$$
\Gamma(2-\alpha)\tau\sum_{n=1}^{m}|\langle f^n,\delta_tU^n\rangle|\leq\tau\sum_{n=1}^{m}\left[\frac{(\Gamma(2-\alpha))^2}{2t_N^{1-\alpha}}|f^{n-\frac{1}{2}}|_D^2+\frac{t_N^{1-\alpha}}{2}|\delta_tU^n|_D^2\right].\tag{3.10}
$$

Again, using (2.3), Cauchy-Schwarz inequality, Young's inequality, and the boundedness of the $\tau^\beta\gamma_n$ and ρ_n

$$
\tau^\beta\gamma_n\leq\tau t_n^{\beta-1},\ \ \rho_n\leq\tau t_n^{\beta-1},\tag{3.11}
$$

(see the result in [4]), we get

$$
\begin{aligned}
&\Gamma(2-\alpha)|\langle q_{\frac{1}{2}}(\Delta U),U^0\rangle|=\Gamma(2-\alpha)\tau^\beta\frac{|\gamma_0|}{2}|\langle\Delta U^1,U^0\rangle|+\Gamma(2-\alpha)(\tau^\beta|\gamma_{\frac{1}{2}}|+|\rho_{\frac{1}{2}}|)|\langle\Delta U^0,U^0\rangle|\\
\leq&\frac{C\Gamma(2-\alpha)\mu}{4}\|\nabla U^1\|^2+\frac{CC(T)^2\Gamma(2-\alpha)}{4\mu}\tau^2\|\nabla U^0\|^2+CC(T)\Gamma(2-\alpha)\tau\|\nabla U^0\|^2,
\end{aligned}\tag{3.12}
$$

(see the proof of (2.22) in [12]), it can be shown that

$$
\begin{aligned}
&\Gamma(2-\alpha)|\langle q_{m-\frac{1}{2}}(\Delta U),U^m\rangle|\leq\Gamma(2-\alpha)\left[\tau^\beta\sum_{p=0}^{m}|\gamma_{m-p}||\langle\Delta U^{p-\frac{1}{2}},U^m\rangle|+|\rho_{m-\frac{1}{2}}||\langle\Delta U^0,U^m\rangle|\right]\\
\leq&C\Gamma(2-\alpha)\tau\sum_{p=0}^{m}\tau^{\beta-1}|\gamma_{m-p}|\|\nabla U^{p-\frac{1}{2}}\|\|\nabla U^m\|+C\Gamma(2-\alpha)|\rho_{m-\frac{1}{2}}|\|\nabla U^0\|\|\nabla U^m\|\\
\leq&C\Gamma(2-\alpha)\left[\tau\sum_{p=0}^{m}\tau^{\beta-1}|\gamma_{m-p}|\left(\frac{\|\nabla U^{p-\frac{1}{2}}\|^2}{2}+\frac{\|\nabla U^m\|^2}{2}\right)+|\rho_{m-\frac{1}{2}}|\left(\frac{\|\nabla U^0\|^2}{2}+\frac{\|\nabla U^m\|^2}{2}\right)\right]\\
\leq&C\Gamma(2-\alpha)\left[\frac{C(T)}{2}\tau\sum_{p=0}^{m}\|\nabla U^{p-\frac{1}{2}}\|^2+\frac{1}{2}\left(\sum_{p=0}^{m}\tau^\beta\gamma_{m-p}+\rho_{m-\frac{1}{2}}\right)\|\nabla U^m\|^2+\frac{C(T)\tau}{2}\|\nabla U^0\|^2\right]\\
\leq&C\Gamma(2-\alpha)\left[\frac{C(T)}{4}\tau\sum_{p=0}^{m}(\|\nabla U^p\|^2+\|\nabla U^{p-1}\|^2)+\frac{C(T)\tau}{2}\|\nabla U^m\|^2+\frac{C(T)\tau}{2}\|\nabla U^0\|^2\right]\\
=&C\Gamma(2-\alpha)\left[\frac{C(T)}{2}\tau\sum_{p=0}^{m-1}\|\nabla U^p\|^2+\frac{3C(T)\tau}{4}\|\nabla U^m\|^2+\frac{C(T)\tau}{2}\|\nabla U^0\|^2\right].
\end{aligned}\tag{3.13}
$$

From (3.11), we have

$$
\sum_{n=1}^{m}\delta_t\rho_{n-\frac{1}{2}}\leq C(T),\tag{3.14}
$$

where $\rho_{-\frac{1}{2}}=0$.

Using (3.14), we obtain

$$\begin{aligned}\Gamma(2-\alpha)\tau\Big|\sum_{n=1}^{m}\langle\delta_t\rho_{n-\frac{1}{2}}\Delta U^0,U^{n-1}\rangle\Big| &\le C\Gamma(2-\alpha)\tau\Big|\sum_{n=1}^{m}\delta_t\rho_{n-\frac{1}{2}}\Big|\|\nabla U^0\|\|\nabla U^{n-1}\| \\ &\le \frac{CC(T)\Gamma(2-\alpha)}{2}\tau\|\nabla U^0\|^2+\frac{CC(T)\Gamma(2-\alpha)}{2}\tau\sum_{n=1}^{m}\|\nabla U^{n-1}\|^2.\end{aligned}\tag{3.15}$$

Also, from (3.11), it is easy to verify that

$$\tau^{\beta}\sum_{p=0}^{n-1}\delta_t\gamma_{p-\frac{1}{2}}\le C(T),\tag{3.16}$$

where $\gamma_{-\frac{1}{2}}=0$.

Applying (3.16), Lemma 3.1, Minkowski inequality and Young's inequality to the second term on the right-hand side of (3.7), it is easy to see that

$$\begin{aligned}&\tau^{\beta+1}\Gamma(2-\alpha)\sum_{n=1}^{m}\sum_{p=0}^{n-1}\delta_t\gamma_{n-p}|\langle\Delta U^{p-\frac{1}{2}},U^{n-1}\rangle|+\tau^{\beta}\Gamma(2-\alpha)|\gamma_0|\sum_{n=1}^{m}|\langle\Delta U^{n-\frac{1}{2}},U^{n-1}\rangle| \\ \le&\tau^{\beta+1}\Gamma(2-\alpha)\sum_{n=1}^{m}\sum_{p=0}^{n-1}\delta_t\gamma_{n-p}\|\nabla U^{p-\frac{1}{2}}\|\|\nabla U^{n-1}\|+\tau^{\beta}\Gamma(2-\alpha)|\gamma_0|\sum_{n=1}^{m}\|\nabla U^{n-\frac{1}{2}}\|\|\nabla U^{n-1}\| \\ \le&C(T)C\Gamma(2-\alpha)\tau\sum_{n=1}^{m}\|\nabla U^{p-1}\|^2+\frac{C(T)C\Gamma(2-\alpha)\tau}{4}\|\nabla U^m\|^2.\end{aligned}\tag{3.17}$$

Combining with the results (3.8)-(3.17), and rearranging the terms, we obtain

$$\begin{aligned}&\frac{C\Gamma(2-\alpha)(\mu-2C(T)\tau)}{2}\|\nabla U^m\|^2\le\frac{C\mu\Gamma(2-\alpha)}{2}\|\nabla U^0\|^2+\frac{CC(T)^2\Gamma(2-\alpha)}{4\mu}\tau^2\|\nabla U^0\|^2 \\ &+2CC(T)\Gamma(2-\alpha)\tau\|\nabla U^0\|^2+\frac{7C(T)C\Gamma(2-\alpha)}{4}\tau\sum_{p=0}^{m-1}\|\nabla U^p\|^2+\frac{(\Gamma(2-\alpha))^2}{2t_N^{1-\alpha}}\tau\sum_{n=1}^{m}|f^{n-\frac{1}{2}}|_D^2 \\ &+\frac{C\Gamma(2-\alpha)\mu}{4}\|\nabla U^1\|^2+\frac{t_N^{2-\alpha}}{2(2-\alpha)}|\psi|_D^2.\end{aligned}\tag{3.18}$$

Applying the discrete Grönwall lemma, the inequality (3.18) implies that

$$\begin{aligned}&\frac{C\Gamma(2-\alpha)(\mu-2C(T)\tau)}{2}\|\nabla U^m\|^2\le\frac{C\mu\Gamma(2-\alpha)}{2}\|\nabla U^0\|^2+\frac{CC(T)^2\Gamma(2-\alpha)}{4\mu}\tau^2\|\nabla U^0\|^2 \\ &+\frac{7CC(T)\Gamma(2-\alpha)}{4}\tau\|\nabla U^0\|^2+\frac{(\Gamma(2-\alpha))^2}{2t_N^{1-\alpha}}\tau\sum_{n=1}^{m}|f^{n-\frac{1}{2}}|_D^2+\frac{C\Gamma(2-\alpha)\mu}{4}\|\nabla U^1\|^2+\frac{t_N^{2-\alpha}}{2(2-\alpha)}|\psi|_D^2.\end{aligned}\tag{3.19}$$

For $m=1$ in (3.19), we have

$$\begin{aligned}&\frac{C\Gamma(2-\alpha)(\mu-4C(T)\tau)}{4}\|\nabla U^1\|^2\le\frac{C\mu\Gamma(2-\alpha)}{2}\|\nabla U^0\|^2+\frac{CC(T)^2\Gamma(2-\alpha)}{4}\tau^2\|\nabla U^0\|^2 \\ &+\frac{7CC(T)\Gamma(2-\alpha)}{4}\tau\|\nabla U^0\|^2+\frac{(\Gamma(2-\alpha))^2}{2t_N^{1-\alpha}}\tau|f^{\frac{1}{2}}|_D^2+\frac{t_N^{2-\alpha}}{2(2-\alpha)}|\psi|_D^2.\end{aligned}\tag{3.20}$$

Combining (3.19), (3.20) with Lemma 2.2, for $\tau<\frac{\mu}{4C(T)}$, we complete the proof. □

Remark: From the stability result in Theorem 3.1, we can see that the unconditional stability is related to the inverse of the space step on the right-hand sides. The similar discussions have been mentioned in [11]. From the subsequent convergence Theorem 4.1, we can observe that this affection would not reduce the global accuracy for regular enough solution. Meanwhile, our numerical tests do not have to satisfy the condition of Theorem 3.1 and remain completely unaffected.

4. Convergence analysis

This section is devoted to deriving abstract error estimates for the proposed Crank-Nicolson ADI OSC numerical scheme. An elliptic projection associated with our numerical scheme will be given first, which is often used in the OSC method research. That is, let $W:[0,T]\to P_r(\delta)$, satisfy [11]

$$\langle \Delta(u-W), V\rangle = 0,\ \forall\, V\in P_r(\delta),\ t>0. \tag{4.1}$$

Then we have the following estimates on $u-W$ and its time derivatives.

Lemma 4.1. [12] *Suppose $\frac{\partial^l u}{\partial t^l}\in H^{r+3-j}$, for $t\in[0,T]$, $l=0,1$, $j=0,1$, and W is defined by equation (4.1), then there exists a constant C, independent of h, such that*

$$\left\|\frac{\partial^l (u-W)}{\partial t^l}\right\|_{H^j} \le Ch^{r+1-j}\left\|\frac{\partial^l u}{\partial t^l}\right\|_{H^{r+3-j}}.$$

Lemma 4.2. [11] *If $\frac{\partial^i u}{\partial t^i}\in H^{r+3}$, for $t\in[0,T]$, $i=0,1$, then we get*

$$\left|\frac{\partial^{l+i}(u-W)}{\partial x^{l_1}\partial y^{l_2}\partial t^i}\right|_D \le Ch^{r+1-l}\left\|\frac{\partial^i u}{\partial t^i}\right\|_{H^{r+3}},$$

where $0\le l=l_1+l_2\le 4$.

Now, we define the error of the second-order fractional convolution quadrature in (1.1) by

$$(R_t^\beta)^n = \frac{1}{\Gamma(\beta)}\int_0^{t_n}(t_n-s)^{\beta-1}\Delta u(x,y,s)ds - \tau^\beta\sum_{p=1}^{n}\gamma_{n-p}\Delta u^p + \rho_n\Delta u^0, \tag{4.2}$$

where $(R_t^\beta)^n$ may be obtained by Lemma 2.1.

Theorem 4.1. *Suppose $U^n\in P_r(\delta)$, $1\le n\le N$ is the solution of the proposed Crank-Nicolson ADI OSC numerical scheme of (2.8) with $U^0=W^0$. If $u(x,y,t)\in C^{4,4,3}_{x,y,t}(\Gamma\times[0,T])$, then there exists a positive constant C, independent of h and τ, such that*

$$\|u(t_n)-U^n\|_{H^j}\le C(h^{r+1-j}+\tau^{3-\alpha}),\ j=0,1,\ 1\le n\le N.$$

Proof. With W defined in (4.1), we write $u(t_n)-U^n=\eta^n-\xi^n$ where $\eta^n=u(t_n)-W^n$ and $\xi^n=U^n-W^n$. Since the estimates for η^n are known from Lemma 4.1 and Lemma 4.2, the main task now is to estimate ξ^n, then we use triangle inequality to bound $\eta^n-\xi^n$.

From (1.1), (2.1) and Lemma 2.2, for $1\le n\le N$, we have

$$\begin{aligned}&\frac{1}{\Gamma(3-\alpha)}\frac{1}{\tau^{\alpha-1}}\left[b_0\delta_t u^n-\sum_{j=1}^{n-1}(b_{n-j-1}-b_{n-j})\delta_t u^j-b_{n-1}\psi\right]+(R_t^\alpha)^{n-\frac12}-\mu\Delta u^{n-\frac12}\\ =&\tau^\beta\sum_{p=0}^{n}\gamma_{n-p}\Delta u^{p-\frac12}+\rho_{n-\frac12}\Delta u^0+(R_t^\beta)^{n-\frac12}+f^{n-\frac12}\ \text{ on }\Gamma.\end{aligned} \tag{4.3}$$

On subtracting (2.7) from (4.3), using (4.1) and (4.2), it follows that

$$
\begin{aligned}
&\frac{1}{\Gamma(3-\alpha)}\frac{1}{\tau^{\alpha-1}}\left[b_0\delta_t\xi^n-\sum_{j=1}^{n-1}(b_{n-j-1}-b_{n-j})\delta_t\xi^j-b_{n-1}\delta_t\eta^1\right]-\mu\Delta\xi^{n-\frac{1}{2}}+\frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\frac{\partial^4\delta_t\xi^n}{\partial x^2\partial y^2}\\
=&\tau^{\beta}\sum_{p=1}^{n}\gamma_{n-p}\Delta\xi^{p-\frac{1}{2}}+\rho_{n-\frac{1}{2}}\Delta\xi^0+T^{n-\frac{1}{2}}\ \text{ on }\ \Gamma,\ \ 1\le n\le N,
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
T^{n-\frac{1}{2}}&=T_1^{n-\frac{1}{2}}+T_2^{n-\frac{1}{2}}+T_3^{n-\frac{1}{2}},\\
T_1^{n-\frac{1}{2}}&=\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\sum_{j=1}^{n-1}b_{n-j-1}(\delta_t\eta^{j+1}-\delta_t\eta^j),\\
T_2^{n-\frac{1}{2}}&=\frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\frac{\partial^4\delta_t\eta^n}{\partial x^2\partial y^2}-\frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\frac{\partial^4\delta_t u^n}{\partial x^2\partial y^2},\\
T_3^{n-\frac{1}{2}}&=-(R_t^{\alpha})^{n-\frac{1}{2}}+(R_t^{\beta})^{n-\frac{1}{2}}.
\end{aligned}
\tag{4.5}
$$

Since (4.4) has the same form as (2.7), following the proof process of the stability in Theorem 3.1, we notice that for $1\le m\le N$,

$$
\|\nabla\xi^m\|^2\le C\|\nabla\xi^0\|^2+C\tau\|\nabla\xi^0\|^2+C\tau^2\|\nabla\xi^0\|^2+Ct_N^{2-\alpha}|\delta_t\eta^1|_D^2+C(T)\tau\sum_{n=1}^{m}|T^{n-\frac{1}{2}}|_D^2. \tag{4.6}
$$

Assume $\xi^0=0$, we get

$$
\|\nabla\xi^m\|^2\le Ct_N^{2-\alpha}|\delta_t\eta^1|_D^2+C(T)\tau\sum_{n=1}^{m}|T^{n-\frac{1}{2}}|_D^2, \tag{4.7}
$$

and

$$
|T^{n-\frac{1}{2}}|_D\le C(T)\tau\sum_{n=1}^{m}\left[|T_1^{n-\frac{1}{2}}|_D+|T_2^{n-\frac{1}{2}}|_D+|T_3^{n-\frac{1}{2}}|_D\right]. \tag{4.8}
$$

Since,

$$
1<t_N^{2-\alpha}<t_1^{2-\alpha}=\tau^{2-\alpha},\ \ 1\le\alpha\le 2,
$$

using Lemma 4.1, the first term on the right-hand side of (4.7) may be bounded by

$$
Ct_N^{2-\alpha}|\delta_t\eta^1|_D^2\le C\tau^{2-\alpha}h^{r+1}. \tag{4.9}
$$

Also, using Lemma 4.1, for $1\le n\le N$, it is easy to see that

$$
\begin{aligned}
|\delta_t^2\eta^{j+1}|_D&=\frac{1}{\tau^2}\left|\int_{t_{j-1}}^{t_j}(s-t_{j-1})\frac{\partial^2\eta}{\partial t^2}(\cdot,s)ds-\int_{t_j}^{t_{j+1}}(s-t_{j+1})\frac{\partial^2\eta}{\partial t^2}(\cdot,s)ds\right|_D\\
&\le\frac{1}{\tau}\left[\int_{t_{j-1}}^{t_j}\left|\frac{\partial^2\eta}{\partial t^2}(\cdot,s)\right|_D ds+\int_{t_j}^{t_{j+1}}\left|\frac{\partial^2\eta}{\partial t^2}(\cdot,s)\right|_D ds\right]\\
&\le Ch^{r+1}\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{(C[0,T],H^{r+3})}.
\end{aligned}
\tag{4.10}
$$

With the help of Lemma 3.2 and (4.10), it follows that

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Big|\sum_{j=1}^{n-1} b_{n-j-1}(\delta_t\eta^{j+1}-\delta_t\eta^j)\Big|_D \leq \frac{\tau^{2-\alpha}}{\Gamma(3-\alpha)}\sum_{j=1}^{n-1} b_{n-j-1}|\delta_t^2\eta^{j+1}|_D. \tag{4.11}$$

Note that, from the notation of b_j in (2.1), we have

$$\sum_{j=0}^{n-2} b_j = (n-1)^{2-\alpha}, \tag{4.12}$$

then applying Lemma 4.2 and (4.12) to (4.11), we obtain

$$|T_1^{n-\frac{1}{2}}|_D \leq C\frac{t_{n-1}^{2-\alpha}}{\Gamma(3-\alpha)}h^{r+1}. \tag{4.13}$$

Writing $T_2^{n-\frac{1}{2}}$ in (4.5) as

$$T_2^{n-\frac{1}{2}} = \frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\frac{\partial^4\delta_t W^n}{\partial x^2\partial y^2} = \frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\Big[\frac{\partial^4\delta_t\eta^n}{\partial x^2\partial y^2} - \frac{\partial^4\delta_t u^n}{\partial x^2\partial y^2}\Big]. \tag{4.14}$$

Thus, using Taylor's theorem and results as in (4.10), together with Lemma 4.1, we get

$$\begin{aligned}|T_2^{n-\frac{1}{2}}|_D &= \frac{\tau\lambda_1(\mu+\lambda_2)^2}{4}\Big|\frac{\partial^4\delta_t W^n}{\partial x^2\partial y^2}\Big|_D \\ &\leq C\tau^{\alpha+1}\Big[\Big\|\frac{\partial u}{\partial t}\Big\|_{C^{2,2,0}} + \Big\|\frac{\partial u}{\partial t}\Big\|_{C([0,T],H^{r+3})}\Big].\end{aligned} \tag{4.15}$$

To bound the term $T_3^{n-\frac{1}{2}}$ in (4.5), using Lemma 2.1, we find that

$$\begin{aligned}\|(R_t^\beta)^n\| &\leq C\tau^2 t_n^{\beta-1}[\|u_{xxt}(x,y,0)\| + \|u_{yyt}(x,y,0)\|] + C\tau^{\beta+1}\int_{t_{n-1}}^{t_n}\big[\|u_{xxtt}(x,y,s)\| + \|u_{yytt}(x,y,s)\|\big]ds \\ &\quad + C\tau^2\int_0^{t_{n-1}}\beta(t_n-s)\big[\|u_{xxtt}(x,y,s)\| + \|u_{yytt}(x,y,s)\|\big]ds \\ &\leq C\tau^2 t_n^{\beta-1} + C\tau^{\beta+1}\int_{t_{n-1}}^{t_n} ds + C\tau^2\int_0^{t_{n-1}}\frac{(t_n-s)^{\beta-1}}{\Gamma(\beta)}ds \\ &\leq C\tau^2 t_n^{\beta-1} + C\tau^{\beta+2} + C\tau^2 t_n^\beta.\end{aligned} \tag{4.16}$$

From the above inequality, we get

$$\begin{aligned}\tau\sum_{n=1}^N \|(R_t^\beta)^n\| &\leq \tau C\sum_{n=1}^N \tau^2 t_n^{\beta-1} + \tau C\sum_{n=1}^N \tau^{\beta+2} + \tau C\sum_{n=1}^N \tau^2 t_n^\beta \\ &\leq C\tau^2\int_0^T s^{\beta-1}ds + C\tau^{2+\beta} + C\tau^2\int_0^T s^\beta ds \\ &\leq C\tau^2.\end{aligned} \tag{4.17}$$

From (2.2), we obtain

$$\tau\sum_{n=1}^N \|(R_t^\alpha)^n\| \leq C\tau^{3-\alpha},\ \ 1\leq\ n\leq N. \tag{4.18}$$

Combine with the results in (4.17) and (4.18), we have

$$\tau\sum_{n=1}^{N}|T_3^{n-\frac{1}{2}}|_D\le C\tau^{3-\alpha}+C\tau^2. \tag{4.19}$$

Substitute (4.13), (4.15) and (4.19) into (4.7), we obtain

$$\tau\sum_{n=1}^{N}|T^{n-\frac{1}{2}}|_D\le Ch^{r+1}+C\tau^{3-\alpha}. \tag{4.20}$$

Then, we obtain

$$\|\nabla\xi^m\|\le Ch^{r+1}+C\tau^{3-\alpha}. \tag{4.21}$$

Thus an application of Poincare's inequality achieves

$$\|\xi^m\|\le Ch^{r+1}+C\tau^{3-\alpha}. \tag{4.22}$$

Finally, combining the triangle inequality, (4.21), (4.22), Lemma 4.1 with the equivalence of the norms on $P_r(\delta)$ in Lemma 2.2 yields the desired estimate. □

5. Numerical experiments

In this section, we carry out numerical experiments for the proposed Crank-Nicolson ADI OSC scheme to illustrate our theoretical analysis. All our tests were done by using MATLAB R2016. We use the space of piecewise Hermite bicubics ($r=3$) with the standard value and scaled slope basis functions. Suppose the step size $h=\frac{L^x}{N_x}=\frac{L^y}{N_y}$, and set $L^x=L^y=\pi$, $T=1$, $\mu=1$. We present the L^2 norm errors and the L^∞ norm errors between the exact and the numerical solutions and the corresponding rates of convergence determined by

$$Rate1=\frac{\log(e_m/e_{m+1})}{\log(h_m/h_{m+1})},\ Rate2=\frac{\log(e_l/e_{l+1})}{\log(\tau_l/\tau_{l+1})}.$$

Example 5.1. *We provide the problem (1.1)-(1.3) with an exact analytical solution*

$$u(x,y,t)=t^{\alpha+\beta+3}\sin(x)\sin(y),$$

with the corresponding forcing term is

$$f(x,y,t)=\left[\frac{\Gamma(\alpha+\beta+4)}{\Gamma(\beta+4)}t^{\beta+3}+2t^{\alpha+\beta+3}+2\frac{\Gamma(\alpha+\beta+4)}{\Gamma(\alpha+2\beta+4)}t^{\alpha+2\beta+3}\right]\sin x\sin y.$$

Table 5.1 represents the L^2 errors and the corresponding convergence order in space, we select the temporal step $\tau\approx h^{(4/\{3-\alpha\})}$ as in [23], $\{(\alpha,\beta)\}=\{(1.15,0.6),(1.5,0.5),(1.75,0.5)\}$. We can observe that the corresponding convergence rates in space for the proposed Crank-Nicolson ADI OSC scheme are approximately 4, which is consistent with the result in Theorem 4.1. Then we set $h=\tau$ to verify the temporal convergence accuracy. Table 5.4 shows the maximum errors and the corresponding convergence order in time. One can see that the convergence orders of the time direction results also match that of the theoretical ones.

Table 5.1: Numerical errors and convergence orders in spatial direction for Example 5.1.

h	$\alpha = 1.15,\ \beta = 0.6$		$\alpha = 1.5,\ \beta{=}0.5$		$\alpha = 1.75,\ \beta = 0.5$	
	L^2error	Rate1	L^2error	Rate1	L^2error	Rate1
1/4	8.9447e-3	$*$	2.2252e-2	$*$	2.8893e-2	$*$
1/8	6.0069e-4	3.8963	1.4027e-3	3.9877	1.8003e-3	3.9996
1/16	4.0477e-5	3.8914	8.8318e-5	3.9894	1.1259e-4	4.0039
1/32	2.6563e-6	3.9296	5.5312e-6	3.9970		
Theory		4.0000		4.0000		4.0000

Table 5.2: Numerical errors and convergence orders in temporal direction for Example 5.1.

τ	$\alpha = 1.15,\ \beta = 0.6$		$\alpha = 1.5,\ \beta = 0.5$		$\alpha = 1.75,\ \beta = 0.5$	
	L^2error	Rate2	L^2error	Rate2	L^2error	Rate2
1/20	9.0791e-3	$*$	6.2185e-2	$*$	1.7361e-1	$*$
1/40	2.6188e-3	1.7836	2.2332e-2	1.4775	7.3027e-2	1.2493
1/80	7.5374e-4	1.7968	7.9782e-3	1.4850	3.0758e-2	1.2475
1/160	2.1622e-4	1.8016	2.8403e-3	1.4900	41.2953e-2	1.2477
Theory		1.8500		1.5000		1.2500

In the second example, we consider a homogeneous case ($f = 0$) for model (1.1), where the exact solution cannot be found readily.

Example 5.2. *In this example, the initial data is chosen as* $u(x, y, 0) = \sin \pi x \sin \pi y,\ x, y \in [0, 1]$, *and the forcing function* $f = 0$.

Table 5.3: Numerical errors and convergence orders in temporal direction for Example 5.2.

τ	$\alpha = 1.5,\ \beta = 0.5$		$\alpha = 1.75,\ \beta = 0.8$		$\alpha = 1.85,\ \beta = 0.75$	
	L^2error	Rate2	L^2error	Rate2	L^2error	Rate2
1/20	1.9651e-2	$*$	1.0447e-1	$*$	1.5286e-1	$*$
1/40	7.0417e-3	1.4806	4.4139e-2	1.2430	7.4805e-2	1.0311
1/80	2.4528e-3	1.5215	1.8545e-2	1.2510	3.5652e-2	1.0692
1/160	8.8697e-4	1.4675	7.8527e-3	1.2398	1.6842e-2	1.0819
Theory		1.5000		1.2500		1.1500

Table 5.4: Numerical errors and convergence orders in temporal direction for Example 5.2.

τ	$\alpha = 1.05,\ \beta = 0.75$		$\alpha = 1.15,\ \beta = 0.95$		$\alpha = 1.25,\ \beta = 0.85$	
	L^2error	Rate2	L^2error	Rate2	L^2error	Rate2
1/20	1.0989e-2	$*$	1.5622e-2	$*$	1.1800e-2	$*$
1/40	5.0827e-3	1.1124	7.3391e-3	1.0900	5.7159e-3	1.0458
1/80	2.4445e-3	1.0560	3.5685e-3	1.0403	2.8082e-3	1.0253
1/160	1.1977e-3	1.0292	1.7610e-3	1.0189	1.3910e-3	1.0135

The numerical results in time direction of Example 5.2 by applying the proposed numerical scheme are listed in Table 5.3 and Table 5.4. We can observe that our method works properly for $\{\alpha =$

1.5, $\beta = 0.5\}$, $\{\alpha = 1.75,\ \beta = 0.8\}$ and $\{\alpha = 1.85,\ \beta = 0.75\}$. However, for $\{\alpha = 1.05,\ \beta = 0.75\}$, $\{\alpha = 1.15,\ \beta = 0.95\}$ and $\{\alpha = 1.25,\ \beta = 0.85\}$, the convergence rate in time direction is one, which shows that the Crank-Nicolson ADI OSC scheme we proposed in this article is only suitable for the exact solution u is smooth enough. We will consider the nonsmooth initial condition for problem (1.1) in our next paper.

Example 5.3. *In order to illustrate the effectiveness of a Crank-Nicolson ADI OSC numerical method, we consider the same problem as in* [22]

$$\begin{cases} D_t^{\alpha}u(x,y,t) - \mu\Delta u(x,y,t) = \left(t^2\frac{\Gamma(\alpha+3)}{2} + 2t^{\alpha+2}\right)\mathrm{sin(x)sin(y)}, & (x,y)\in\Omega,\quad t\in(0,1], \\ u(x,y,0) = \phi(x,y),\quad u_t(x,y,0) = \psi(x,y), & (x,y)\in\overline{\Omega} = \Omega\cup\partial\Omega, \\ u(x,y,t) = 0,\quad (x,y)\in\partial\Omega,\quad t\in(0,T], \end{cases} \tag{5.1}$$

where $\Omega = (0,\pi)\times(0,\pi)$.

The exact solution to (5.1) is $u(x,y,t) = t^{\alpha+2}\mathrm{sin(x)sin(y)}$. We select the same time and space variables as in [22], Table 5.5, 5.6 and 5.7 display L^∞ errors for different α in time, from these Tables one can easily find that the present Crank-Nicolson ADI OSC method shows better performance than that in [22] for this example. It proves the validity of the OSC method.

Table 5.5. Max errors and convergence rates in time when $h = k$.

N	$M_1 = M_2$	$\alpha = 1.25$	CPU(s)	$\alpha = 1.25$ [22]	CPU(s)
24	4	1.8288-003	0.0043	2.1782e-003	0.0470
60	6	3.7803-004	0.0378	4.5846e-004	0.4220
116	8	1.2063-004	0.0539	1.4733e-004	1.7190
193	10	4.9805-005	0.1485	6.0965e-005	5.6250

Table 5.6. Max errors and convergence rates in time when $h = k$.

N	$M_1 = M_2$	$\alpha = 1.5$	CPU(s)	$\alpha = 1.5$ [22]	CPU(s)
40	4	3.7261-003	0.0161	3.9333e-003	0.1100
118	6	7.4703-004	0.0447	8.0156e-004	0.9060
256	8	2.3492-004	0.1385	2.5323e-004	5.0780
464	10	9.6481-005	0.4387	1.0412e-004	20.7810

Table 5.7. Max errors and convergence rates in time when $h = k$.

N	$M_1 = M_2$	$\alpha = 1.75$	CPU(s)	$\alpha = 1.75$ [22]	CPU(s)
84	4	5.5925-003	0.0241	5.7572e-003	0.2340
309	6	1.1132-003	0.1491	1.1535e-003	3.5320
776	8	3.5378-004	0.7501	3.6654e-004	30.2970
1585	10	1.4519-004	3.5081	1.5035e-004	178.5930

6. Conclusions

In this paper, we construct a Crank-Nicolson ADI OSC scheme for the 2D fractional integro-differential equation, and give strict stability and convergence analysis. The theoretical analysis has been verified by some numerical results. To the best of author's knowledge, however, few numerical methods have been derived so far for solving problem (1.1)-(1.3). In future work, we plan to apply the OSC method to nonsmooth initial condition for problem (1.1).

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